The 7th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 9th April 1997
Category II

Problem 1. Decide whether it is possible to cover the 3-dimensional Euclidean space with lines which are pairwise skew (i.e. not coplanar).
[12 points]

Problem 2. Let $f\colon \mathbb{C} \to \mathbb{C}$ be a holomorphic function with the property that $\bigl|f(z)\bigr| = 1$ for all $z \in \mathbb{C}$ such that $|z| = 1$. Prove that there exist a $\theta \in \mathbb{R}$ and a $k \in \{0, 1, 2, \ldots\}$ so that

$$f(z) = \mathrm{e}^{\mathrm{i}\theta} z^k$$

for all $z \in \mathbb{C}$. [10 points]

Problem 3. Let $u \in C^2\bigl(\overline{D}\bigr)$, $u = 0$ on ∂D where D is the open unit ball in $\mathbb{R}^3$. Prove that the following inequality holds for all $\varepsilon > 0$:

$$\int_D |\nabla u|^2 \,\mathrm{d}V \le \varepsilon \int_D (\Delta u)^2 \,\mathrm{d}V + \frac{1}{4\varepsilon} \int_D u^2 \,\mathrm{d}V .$$

(We recall that $\nabla u = \left[\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right]$ and $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$ are gradient and Laplacian respectively.) [13 points]

Problem 4-M. Prove that

$$\sum_{n=1}^{\infty} \frac{n^2}{(7n)!} = \frac{1}{7^3} \sum_{k=1}^{2} \sum_{j=0}^{6} \mathrm{e}^{\cos(2\pi j/7)} \cdot \cos\left(\frac{2k\pi j}{7} + \sin\frac{2\pi j}{7}\right).$$

[15 points]

Problem 4-I. Problem *Div*$_3$ is specified as follows:

Instance: a finite string of symbols 0 and 1.
Question: is the given string a binary code of a number divisible by 3?

(It is obvious that there is a program which solves problem *Div*$_3$, i.e., it outputs the right answer *yes* or *no* for any string of 0's and 1's.)

But you should show that there is no program *Gen-Test-Data* with the specification:

Input: any program P.
Output: a finite set $D(P)$ of strings of 0's and 1's such that the program P solves problem *Div*$_3$ iff the program P outputs the correct answer for all inputs from $D(P)$.

Remark. You can use the Recursion Theorem, which can be expressed in the following form:

For any program Transf which transforms programs in some way (i.e., for any given program P, it constructs some other program $P' =$ Transf(P)), there exists a program P_0 whose input/output behaviour is not affected by the transformation (i.e., P_0 and Transf(P_0) yield the same outputs for the same inputs). [15 points]