The 3$^{\text{rd}}$ Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 13$^{\text{th}}$-14$^{\text{th}}$ April 1993
Category I

Problem 1 *Decide whether there is a nontrivial homomorphism from the additive group of rational numbers to the additive group of integers.*

Solution (by Stijn Cambie) Suppose it is possible to have a non-trivial homomorphism ϕ. In that case there is some rational q such that $\phi(q) = n$ for some non-zero integer n. By additivity we have $\phi(2n\frac{q}{2n}) = 2n\phi(\frac{q}{2n}))$. Hence $\phi(\frac{q}{n}) = \frac{1}{2}$ and as $\frac{1}{2}$ is not an integer, we have a contradiction. Hence such a homomorphism does not exist. □

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Problem 2 *Let A be a real magic matrix, i.e. there exists a nonzero real number S such that the sum of each row is equal to S, the sum of each column is equal to S, the sum of the elements of the main diagonal is equal to S and the sum of the elements of the secondary diagonal is equal to S.*

1. *Prove that if A is invertible then A^{-1} is magic.*
2. *Show that*
$$A = \begin{pmatrix} \frac{S}{3}+u & \frac{S}{3}-u+v & \frac{S}{3}-v \\ \frac{S}{3}-u-v & \frac{S}{3} & \frac{S}{3}+u+v \\ \frac{S}{3}+v & \frac{S}{3}+u-v & \frac{S}{3}-u \end{pmatrix},$$
where u and v are arbitrary numbers. Further show that A is not singular if and only if $u^2 \neq v^2$.

Solution First we prove that

$$A = \begin{pmatrix} \frac{S}{3}+u & \frac{S}{3}-u+v & \frac{S}{3}-v \\ \frac{S}{3}-u-v & \frac{S}{3} & \frac{S}{3}+u+v \\ \frac{S}{3}+v & \frac{S}{3}+u-v & \frac{S}{3}-u \end{pmatrix}.$$

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

be a magic matrix. From this we get

$$a_{22} = S - a_{11} - a_{33}\,, \tag{1}$$
$$a_{22} = S - a_{13} - a_{31}\,, \tag{2}$$
$$a_{22} = S - a_{12} - a_{32}\,, \tag{3}$$
$$a_{22} = S - a_{21} - a_{23}\,. \tag{4}$$

Hence $a_{22} = \frac{S}{3}$. From (1) we obtain $a_{11} = \frac{S}{3}+u$ and $a_{33} = \frac{S}{3}-u$. From (2) we get $a_{31} = \frac{S}{3}+v$ and $a_{13} = \frac{S}{3}-v$. From $a_{11}+a_{12}+a_{13} = S$ we get $a_{12} = \frac{S}{3}-u+v$ and $a_{11}+a_{21}+a_{31} = S$ we obtain $a_{21} = \frac{S}{3}-u-v$. From (3) and (4) we get $a_{32} = \frac{S}{3}+u-v$ and $a_{23} = \frac{S}{3}+u+v$. We have matrix A in the form

$$A = \begin{pmatrix} \frac{S}{3}+u & \frac{S}{3}-u+v & \frac{S}{3}-v \\ \frac{S}{3}-u-v & \frac{S}{3} & \frac{S}{3}+u+v \\ \frac{S}{3}+v & \frac{S}{3}+u-v & \frac{S}{3}-u \end{pmatrix}.$$

The determinant of the matrix A is $3S(v^2-u^2)$, so if $v^2 = u^2$ then the matrix A is singular; otherwise is regular.

Now suppose that A is regular. Then

$$A^{-1} = \begin{pmatrix} -\frac{u\,S-v^2+u^2}{(3\,v^2-3\,u^2)\,S} & -\frac{S-v-u}{(3\,v+3\,u)\,S} & \frac{v\,S+v^2-u^2}{(3\,v^2-3\,u^2)\,S} \\ \frac{S+v-u}{(3\,v-3\,u)\,S} & \frac{1}{3\,S} & -\frac{S-v+u}{(3\,v-3\,u)\,S} \\ -\frac{v\,S-v^2+u^2}{(3\,v^2-3\,u^2)\,S} & \frac{S+v+u}{(3\,v+3\,u)\,S} & \frac{u\,S+v^2-u^2}{(3\,v^2-3\,u^2)\,S} \end{pmatrix}$$

and it is easy to check that this matrix is also magic. □

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Problem 3 *Does there exist an injective function $f\colon \mathbb{R} \to \mathbb{R}$ satisfying the inequality*

$$f(x^2) - (f(x))^2 \geq \frac{1}{4}$$

for all $x \in \mathbb{R}$?

Solution Let $f\colon \mathbb{R} \to \mathbb{R}$ be an injective function such that

$$f(x^2) - (f(x))^2 \geq \frac{1}{4}$$

for all $x \in \mathbb{R}$. For $x = 0$, we have

$$\begin{aligned} f(0) - (f(0))^2 &\geq \frac{1}{4} \\ (f(0))^2 - f(0) + \frac{1}{4} &\leq 0 \\ \left(f(0) - \frac{1}{2}\right)^2 &\leq 0 \end{aligned}$$

and from this we get $f(0) = \frac{1}{2}$.

But on the other hand for $x = 1$ we get

$$\begin{aligned} f(1) - (f(1))^2 &\geq \frac{1}{4} \\ (f(1))^2 - f(1) + \frac{1}{4} &\leq 0 \\ \left(f(1) - \frac{1}{2}\right)^2 &\leq 0 \end{aligned}$$

and from this we get $f(1) = \frac{1}{2}$. Hence $f(0) = f(1) = \frac{1}{2}$ and because the function f is an injective function we get a contradiction, so such a function does not exists. $\square$

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Problem 4 *Let* $a_0 = 6^{1992}, a_1 = 3 \cdot 6^{1991}, \ldots$ *be a geometric progression and* $b_0 = 465 \cdot 3^{1985}, b_1 = 466 \cdot 3^{1985}, b_2 = 467 \cdot 3^{1985}, \ldots$ *be an arithmetic progression. Find* n *such that* $a_n = b_n$.

Solution We have $a_n = 3^n \cdot 6^{1992-n}$ and $b_n = (465+n) \cdot 3^{1985}$. We solve

$$\begin{aligned} 3^n \cdot 6^{1992-n} &= (465+n) \cdot 3^{1985} \\ 3^{1992} \cdot 2^{1992-n} &= (465+n) \cdot 3^{1985} \\ 3^7 \cdot 2^{1992-n} &= 465+n\,. \end{aligned} \tag{1}$$

For $n > 1992$ the term $3^7 \cdot 2^{1992-n}$ is not an integer and equation (1) does not hold for any such n. Hence we have two possibilities.

1. For $n = 1992$ we get $3^7 < 465 + 1992$.

2. For $n < 1992$ we get $3^7 \cdot 2^{1992-n} > 465 + n$.

Hence we cannot find $n \in \mathbb{N}$ such that $a_n = b_n$. □

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Category II

Problem 1 *Decide if*

1. $Q[x]/(x^2 - 1) \simeq Q[x]/(x^2 - 4)$
2. $Q[x]/(x^2 + 1) \simeq Q[x]/(x^2 + 2x + 2)$,

where $Q[x]$ *is the ring of polynomials with rational coefficients and* $(f(x))$ *is the prime ideal in* $Q[x]$ *generated by* $f(x)$.

The 3rd Annual Vojtěch Jarník
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Problem 2 *Let* $n \geq 1$ *be and* m_i *be natural numbers such that* $m_i < p_{n-i}$ $(0 \leq i \leq n-1)$, *where* p_k *is kth-prime. Prove that if* $m_0/p_n + \ldots + m_{n-1}/2$ *is a natural number then* $m_0 = \ldots = m_{n-1} = 0$.

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Problem 3 *Let $P^{(4)}(x) = x^6 + x^2 + 1$. Prove that $P(x)$ does not have ten distinct roots.*

Solution The polynomial $P(x)$ has at least ten roots then the polynomial $P'(x)$ has at least nine roots, so $P^{(4)}(x)$ has at least six roots. $P^{(4)}(x) = x^6 + x^2 + 1$ and after substitution $x^2 = y$ we get the polynomial $H(y) = y^3 + y + 1$. Because $H'(y) = 3y^2 + 1$ does not have real roots, we obtain that $P(x)$ does not have ten real roots. □

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Problem 4 *Prove that if $f\colon \mathbb{R} \to \mathbb{R}$ fulfill the inequalities*

$$f(x) \le x\,, \qquad f(x+y) \le f(x) + f(y)$$

for all $x, y \in \mathbb{R}$, then $f(x) = x$ for all $x \in \mathbb{R}$.

Solution Let $\exists\, x \in \mathbb{R}\colon f(x) < x$. Hence

$$f(0) = f(x-x) \le f(x) + f(-x) < x + f(-x)\,.$$

And because $f(-x) \le -x$ we have

$$f(0) < x - x \quad \Rightarrow \quad f(0) < 0\,.$$

Hence

$$f(x) = f(0+x) \le f(0) + f(x) < 0 + f(x)\,.$$

We obtained $f(x) < f(x)$ which is contradiction, so $f(x) = x$ for all $x \in \mathbb{R}$. $\square$