The 3^{rd} Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 13^{th} April 1993
Category I

Problem 1 *Decide whether there is a nontrivial homomorphism from the additive group of rational numbers to the additive group of integers.*

Problem 2 *Let A be a real magic matrix, i.e. there exists a nonzero real number S such that the sum of each row is equal to S, the sum of each column is equal to S, the sum of the elements of the main diagonal is equal to S and the sum of the elements of the secondary diagonal is equal to S.*

1. *Prove that if A is invertible then A^{-1} is magic.*
2. *Show that*
$$A = \begin{pmatrix} \frac{S}{3} + u & \frac{S}{3} - u + v & \frac{S}{3} - v \\ \frac{S}{3} - u - v & \frac{S}{3} & \frac{S}{3} + u + v \\ \frac{S}{3} + v & \frac{S}{3} + u - v & \frac{S}{3} - u \end{pmatrix},$$
where u and v are arbitrary numbers. Further show that A is not singular if and only if $u^2 \neq v^2$.

Problem 3 *Does there exist an injective function $f\colon \mathbb{R} \to \mathbb{R}$ satisfying the inequality*
$$f(x^2) - (f(x))^2 \geq \frac{1}{4}$$
for all $x \in \mathbb{R}$?

Problem 4 *Let $a_0 = 6^{1992}, a_1 = 3 \cdot 6^{1991}, \ldots$ be a geometric progression and $b_0 = 465 \cdot 3^{1985}, b_1 = 466 \cdot 3^{1985}, b_2 = 467 \cdot 3^{1985}, \ldots$ be an arithmetic progression. Find n such that $a_n = b_n$.*

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Problem 1 *Decide if*

1. $Q[x]/(x^2-1) \simeq Q[x]/(x^2-4)$
2. $Q[x]/(x^2+1) \simeq Q[x]/(x^2+2x+2)$,

where $Q[x]$ *is the ring of polynomials with rational coefficients and* $(f(x))$ *is the prime ideal in* $Q[x]$ *generated by* $f(x)$.

Problem 2 *Let* $n \geq 1$ *be and* m_i *be natural numbers such that* $m_i < p_{n-i}$ $(0 \leq i \leq n-1)$, *where* p_k *is kth-prime. Prove that if* $m_0/p_n + \ldots + m_{n-1}/2$ *is a natural number then* $m_0 = \ldots = m_{n-1} = 0$.

Problem 3 *Let* $P^{(4)}(x) = x^6 + x^2 + 1$. *Prove that* $P(x)$ *does not have ten distinct roots.*

Problem 4 *Prove that if* $f\colon \mathbb{R} \to \mathbb{R}$ *fulfill the inequalities*

$$f(x) \leq x\,, \qquad f(x+y) \leq f(x) + f(y)$$

for all $x, y \in \mathbb{R}$, *then* $f(x) = x$ *for all* $x \in \mathbb{R}$.