The $2^{\text{nd}}$ Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, $28^{\text{th}}$ April 1992
Category I

**Problem 1** *Find the $n^{\text{th}}$ derivation of the function*

$$f(x) = \frac{x}{x^2-1}\,.$$

**Solution**

$$\begin{aligned} f(x)^{(n)} &= \Big(\frac{x}{x^2-1}\Big)^{(n)} = \frac{1}{2}\Big(\frac{1}{x+1} + \frac{1}{x-1}\Big)^{(n)} \\ &= \frac{(-1)^n}{2} n!\Big(\frac{1}{(x+1)^{n+1}} + \frac{1}{(x-1)^{n+1}}\Big)\,. \end{aligned}$$

□

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**Problem 2** *Prove that there exist two real convex functions $f, g$ such that*

$$f(x) - g(x) = \sin x$$

*for all $x \in \mathbb{R}$.*

**Solution** Let $f(x) = x^2 + \sin x$ and $g(x) = x^2$, then $f(x) - g(x) = \sin x$ and $f(x), g(x)$ are convex function because

$$f''(x) = 2 - \sin x > 0\,,$$
$$g''(x) = 2 > 0\,.$$

□

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**Problem 3** *Prove that for all integers $n > 1$,*

$$(n-1)|(n^n - n^2 + n - 1)\,.$$

**Solution** For $n = 2$ we get $(n-1) = 1$ and $(n^n - n^2 + n - 1) = 1$ so $(n-1)|(n^n - n^2 + n - 1)$.

Now we prove the result for $n \geq 3$. First we use mathematical induction to prove that $(n-1)|(n^k - n^2)$ for $k \geq 3$. For $k = 3$ we have

$$(n^3 - n^2) : (n-1) = n^2 \quad \Rightarrow \quad (n-1)|(n^3 - n^2)\,.$$

Suppose that $(n-1)|(n^k - n^2)$. We have to prove that $(n-1)|(n^{k+1} - n^2)$. Hence $(n^{k+1} - n^2)/(n-1) = n^k$ with remainder $(n^k - n^2)$, and because $n-1$ divides $(n^k - n^2)$ we have shown that $(n-1)|(n^k - n^2)$ for $k \geq 3$. Hence from the facts that $(n-1)|(n^n - n^2)$ and $(n-1)|(n-1)$ we obtain that $(n-1)|(n^n - n^2 + n - 1)$ for all $n > 1$. □

**Second solution** Let $n > 1$ be a fixed integer. Let $f(x)$ be the polynomial $x^n - x^2 + x - 1$. Note that $f(1) = 0$. It follows from the factor theorem that $(x-1)|f(x)$. Substituting in n for x and noting that $n - 1 \neq 0$, we see that

$$n - 1|f(n) = n^n - n^2 + n - 1\,.$$

□

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**Problem 4** *Let $X$ be a finite set and $f\colon X \to X$ be map. Prove that $f$ is an injective map if and only if $f$ is a surjective map.*

**Solution** Let $X$ have $n$ elements.

1. Let $f$ be injective, i.e. $\forall x_i, x_j \in X; f(x_i) \neq f(x_j), i \neq j$. Then we have $\forall x_i \in X, f(x_i) = y_i \in X$ and because $f$ is injective then $y_i \neq y_j$ whenever $i \neq j$. Because $f\colon X \to X$ we get $f$ is surjective.

2. Suppose $f$ is surjective, but it is not injective. Then $\exists x_i, x_j$ such that $f(x_i) = f(x_j)$. But because $f\colon X \to X$ we obtain that $f$ is not surjective. This is contradiction.

$\square$

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Category II

**Problem 1** *Prove that for a continuously differentiable function* $f(x)$, *where* $f(a) = f(b) = 0$,

$$\max_{x \in [a,b]} |f'(x)| \geq \frac{1}{(b-a)^2} \int_a^b |f(x)| \, \mathrm{d}x \, .$$

**Solution**

$$\begin{aligned} \frac{1}{(b-a)^2} \int_a^b |f(x)| \, \mathrm{d}x &\leq \frac{1}{(b-a)^2} \max_{x \in [a,b]} |f(x)| \int_a^b 1 = \frac{\max_{x \in [a,b]} |f(x)|}{b-a} \\ &= \frac{\max_{x \in [a,b]} |f(x) - f(a)|}{b-a} = \frac{\max_{x \in [a,b]} |(x-a) f'(\xi)|}{b-a} \\ &\leq \max_{\xi \in [a,b]} |f'(\xi)| \, . \end{aligned}$$

□

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Category II

**Problem 2** *Find all functions* $f\colon \mathbb{R} \to \mathbb{R}$ *which satisfy the equality*

$$xf(y) + yf(x) = (x + y)f(x)f(y)\,.$$

**Solution** For $x = y = 1$ we get $f(1) = f^2(1)$. Hence we have two possibilities. For $f(1) = 0, y = 1$ we obtain that $f(x) = 0$ and for $f(1) = 1, y = 1$ we have

$$f(x) = \begin{cases} 1 & x \neq 0 \\ a \in \mathbb{R} & x = 0 \end{cases}\,.$$

□

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Category II

**Problem 3** *Let $Z_k$ be the additive group of residual classes modulo $k$. Decide if $Z_6$ is isomorphic to $Z_2 \times Z_3$.*

**Solution** Let $Z_6$ be

| $\oplus$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 |
| 1 | 1 | 2 | 3 | 4 | 5 | 0 |
| 2 | 2 | 3 | 4 | 5 | 0 | 1 |
| 3 | 3 | 4 | 5 | 0 | 1 | 2 |
| 4 | 4 | 5 | 0 | 1 | 2 | 3 |
| 5 | 5 | 0 | 1 | 2 | 3 | 4 |

and $Z_2 \times Z_3$

| $\boxplus$ | (0,0) | (0,1) | (0,2) | (1,0) | (1,1) | (1,2) |
|---|---|---|---|---|---|---|
| (0,0) | (0,0) | (0,1) | (0,2) | (1,0) | (1,1) | (1,2) |
| (0,1) | (0,1) | (0,2) | (0,0) | (1,1) | (1,2) | (1,0) |
| (0,2) | (0,2) | (0,0) | (0,1) | (1,2) | (1,0) | (1,1) |
| (1,0) | (1,0) | (1,1) | (1,2) | (0,0) | (0,1) | (0,2) |
| (1,1) | (1,1) | (1,2) | (1,0) | (0,1) | (0,2) | (0,0) |
| (1,2) | (1,2) | (1,0) | (1,1) | (0,2) | (0,0) | (0,1) |

Let $f\colon Z_6 \to Z_2 \times Z_3$ be the function defined by

$$f(0) = (0,0), \quad f(1) = (1,1), \quad f(2) = (0,2),$$
$$f(3) = (1,0), \quad f(4) = (0,1), \quad f(5) = (1,2)$$

It is easy to check that this function is an injective function and the condition

$$f(x \oplus y) = f(x) \boxplus f(y)$$

is fulfilled. Thus $Z_6$ is isomorphic to $Z_2 \times Z_3$. □

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**Problem 4** *Prove that each rational number* $\frac{p}{q} \neq 0$ *can be written in the form*

$$\frac{p}{q} = b_1 + \frac{b_2}{2!} + \cdots + \frac{b_n}{n!} ,$$

*where* $n$ *is a sufficiently large positive integer and* $b_k \in \mathbb{Z}$ $(k > 1)$ *such that* $0 \leq b_k < k,\ b_n \neq 0.$

**Solution** Let

$$\frac{p}{q} = \frac{p(q-1)!}{q!} .$$

Further we have

$$\begin{aligned} p(q-1)! &= s_q q + b_q , \\ s_q &= s_{q-1}(q-1) + b_{q-1} , \\ &\vdots \\ s_1 &= b_1 . \end{aligned}$$

where $b_i \in \{0, \ldots, i-1\}$. So we obtain

$$\frac{p}{q} = b_1 + \frac{b_2}{2!} + \cdots + \frac{b_q}{q!} .$$

□