The 1st Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 10th April 1991
Category I

Problem 1 *Show that there is no arithmetic progression with three elements in the infinite geometric sequence* $\{2^k\}_{k=0}^{\infty}$.

Solution Suppose $2^k, 2^l, 2^m$ form an arithmetic progression, where $0 < k < l < m$. Then $2^l - 2^k = 2^m - 2^l$ and therefore $2^{l+1} = 2^k(2^{m-k} + 1)$. Because of $k < l < m$, $m \geq k + 2$, $2^{m-k} + 1$ is an odd number greater or equal to 5. This odd number must divide 2^{l+1}. This is a contradiction. $\square$

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Problem 2 *Compute the determinant*

$$\det \begin{pmatrix} 0 & a_{12} & a_{13} & \dots & a_{1n} \\ -a_{12} & 0 & a_{23} & \dots & a_{2n} \\ -a_{13} & -a_{23} & 0 & \dots & a_{3n} \\ \dots & & & & \\ -a_{1n} & -a_{2n} & -a_{3n} & \dots & 0 \end{pmatrix},$$

where n is an odd number.

Solution Let

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} & \dots & a_{1n} \\ -a_{12} & 0 & a_{23} & \dots & a_{2n} \\ -a_{13} & -a_{23} & 0 & \dots & a_{3n} \\ \dots & & & & \\ -a_{1n} & -a_{2n} & -a_{3n} & \dots & 0 \end{pmatrix}$$

and $d = \det A$. If we multiply all rows in A by -1, we get new determinant d'. The following holds:

$$d' = (-1)^n d = -d.$$

On the other hand,

$$d' = \det A^T,$$

where A^T is the transposed matrix of A. Thus

$$-d = d' = \det A^T = \det A = d.$$

This implies that $d = 0$. $\square$

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Problem 3 *Let $[x]$ be the integer part of x. Find the limit*

$$\lim_{n\to\infty}\big((\sqrt{3}+1)^n-[(\sqrt{3}+1)^n]\big).$$

Solution For n even we have

$$\begin{aligned}(\sqrt{3}+1)^{2n}-[(\sqrt{3}+1)^{2n}]&=(\sqrt{3}+1)^{2n}+(\sqrt{3}-1)^{2n}-(\sqrt{3}-1)^{2n}-[(\sqrt{3}+1)^{2n}]\\&=1-(\sqrt{3}-1)^{2n}\,,\end{aligned}$$

but for n odd we get

$$\begin{aligned}(\sqrt{3}+1)^{2n+1}-[(\sqrt{3}+1)^{2n+1}]&=(\sqrt{3}+1)^{2n+1}-(\sqrt{3}-1)^{2n+1}\\&\quad+(\sqrt{3}-1)^{2n+1}-[(\sqrt{3}+1)^{2n+1}]\\&=(\sqrt{3}-1)^{2n+1}\end{aligned}$$

So the limit does not exists.

□

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Problem 4 *Let $f(x)$ be an even, twice continuously differentiable function and $f''(0) \neq 0$. Prove that there is an extremum of $f(x)$ at the point $x = 0$.*

Solution

$$\lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = f'(0)\,,$$

and because $f(h) = f(-h)$ we have

$$\lim_{h \to 0^+} \frac{f(h) - f(0)}{h} + \lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^+} \left(\frac{f(h) - f(0)}{h} + \frac{f(-h) - f(0)}{-h} \right) = 0\,,$$

which implies that $2f'(0) = 0$. Hence $f'(0) = 0$ and because $f''(0) \neq 0$ we obtain that $f(x)$ has an extremum at the point $x = 0$. $\square$